Lecture 21

First thank everyone for your suggestions given in the exam!

Some common suggestions:

- Don't scroll up and down the note during lecture.

Thanks for letting me know. will avoid doing it.

- Give Review class before the final.

Yes, we will!

- Give more choices of OH to make it more flexible

New OH schedule:

Mondays: 4:30pm - 6:00 pm

Thursdays: ~~5pm - 6pm~~ should be "4pm-5pm"

Fridays: ~~5pm-6pm~~: should be "4pm-5pm"

- Other suggestions

Thanks! Try to accommodate

---

plan of today: § 8.2 power series

Some definitions from Calculus:

- Infinite sum: $\sum_{n=0}^{\infty} b_n$ with $b_n \in \mathbb{R}$

we say $\sum_{n=0}^{\infty} b_n$ converges if the limit

$\lim_{N\to\infty} \sum_{n=0}^{N} b_n = L$ exists (In this case, we write $\sum_{n=0}^{\infty} b_n = L$)

mean exists as a finite number ($L \in \mathbb{R}$, $L \neq \infty, -\infty$)

otherwise, we say $\sum_{n=0}^{\infty} b_n$ diverges

- power Series

A power series about a pt x_0 $\in \mathbb{R}$ means

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n \quad \left(= \lim_{N\to\infty} \sum_{n=0}^{N} a_n (x-x_0)^n \right)$$

$$= \underbrace{a_0}_{n=0} + \underbrace{a_1(x-x_0)}_{n=1} + \underbrace{a_2(x-x_0)^2}_{n=2} + \cdots$$

Here x is a variable, a_n's are constants.

- we say the series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges at $x=c$ if $\sum_{n=0}^{\infty} a_n (c-x_0)^n$ converges.

- otherwise, we say $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ diverges at $x=c$.

- we say the series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges "absolutely" at $x=c$ if
$$\sum_{n=0}^{\infty} |a_n(c-x_0)^n| \text{ converges}$$

Remark:

"Converges absolutely" $\Rightarrow$ "converges"

Why? You will learn the proof in Math 142/140.

Recall again from Calculus.

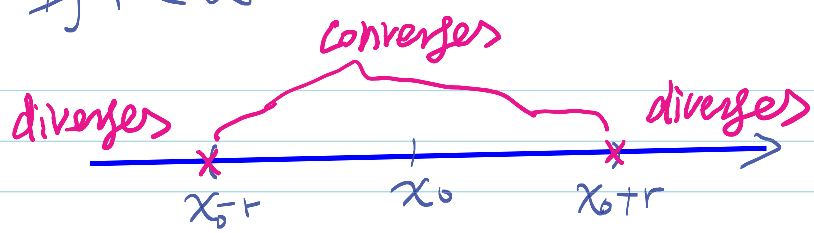
Fact: Given a power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$,
there is r (either nonnegative real number or "∞")
such that
Called radius of convergence! "r" is unique

the series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges if $|x-x_0|<r$

and diverges if $|x-x_0|>r$.

Note: If $r=\infty$, then it means $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges for all $x \in \mathbb{R}$.

- If $r < \infty$



Q: How about the two end pts $x_0 - r$, $x_0 + r$?

A: depends. Any case can happen.

Q: How to find the radius of convergence r ?

A: Root test and Ratio test.

Given $\sum_{n=0}^{\infty} a_n (x - x_0)^n$,

Root test: radius of convergence $r = \dfrac{1}{\lim_{n\to\infty} \sqrt[n]{|a_n|}}$

if the limit $\lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists

Ratio test: radius of convergence $r = \lim_{n\to\infty} \left| \dfrac{a_n}{a_{n+1}} \right|$

if the limit $\lim_{n\to\infty} \left| \dfrac{a_n}{a_{n+1}} \right|$ exists.

E.g: Determine the radius of convergence r of

$$\sum_{n=0}^{\infty} \underbrace{\frac{(-2)^n}{n+1}}_{a_n} (x-\underbrace{3}_{x_0=3})^n$$

$r = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$

A: We use ratio test:

Note $a_n = \frac{(-2)^n}{n+1}$ (1)

Next find a_{n+1}. (How? Replace "n" by "n+1" in (1))

$$\Rightarrow \quad a_{n+1} = \frac{(-2)^{n+1}}{n+2}$$

Hence $\left|\frac{a_n}{a_{n+1}}\right| = |a_n| \cdot \left|\frac{1}{a_{n+1}}\right|$

$$= \frac{2^n}{n+1} \cdot \frac{n+2}{2^{n+1}}$$

$$= \frac{1}{2} \frac{n+2}{n+1}$$

$$\Rightarrow \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{1}{2} \frac{n+2}{n+1}$$

$$= \frac{1}{2} \lim_{n\to\infty} \frac{n+2}{n+1} = \frac{1}{2}$$

Hence the radius of convergence $r = \frac{1}{2}$.

Hint for Q3 in HW 7: $\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n^2}$

Rewrite it as $\sum_{n=0}^{\infty} \frac{2^n}{n^2}\left(x+\frac{1}{2}\right)^n$

Note $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ defines a function on the interval (x_0-r, x_0+r): (r = radius of convergence)

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n \quad \text{on } (x_0-r, x_0+r).$$

And if a function $f(x)$ equals to $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ on some interval containing x_0, then we say

f has Taylor expansion $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ at x_0.

E.g.:

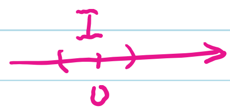
$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{on } (-1, 1).$$

$x_0 = 0$, $r = 1$

The RHS is called the geometric series.

Thm: (Vanishing series)

If $\sum_{n=0}^{\infty} a_n (x-x_0)^n = 0$ for all x in some open interval containing 0, then all $a_n = 0$.



Some important operations of power series:

In the following, we will focus on the case $x_0 = 0$.

- Sum of two power series:

Assume $\begin{cases} f(x) = \sum_{n=0}^{\infty} a_n x^n \\ g(x) = \sum_{n=0}^{\infty} b_n x^n \end{cases}$ on $I = (-r, r)$

Then

$$f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n$$

on I

product of two power series

Assume
$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{on } I=(-r,r).$$
$$g(x) = \sum_{n=0}^{\infty} b_n x^n$$

Then
$$f(x)\,g(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right)$$
$$= (a_0 b_0) + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots$$

Why?
$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$
$$g(x) = \sum_{n=0}^{\infty} b_n x^n = b_0 + b_1 x + b_2 x^2 + \cdots$$

Hence $f(x)\,g(x)$

Higher order

Higher order
$$= (a_0 + a_1 x + a_2 x^2 + \cdots)(b_0 + b_1 x + b_2 x^2 + \cdots)$$

Let's compute by the type of terms in $f(x)g(x)$.

Constant: $a_0 b_0$

x: $a_0(b_1 x) + (a_1 x) b_0 = (a_0 b_1 + a_1 b_0) x$

x^2: $a_0(b_2 x^2) + (a_1 x)(b_1 x) + (a_2 x^2) b_0$
$= (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2$

$a_0 \cdot x^0 \; b_2 \cdot x^2$
In general
$(a_k x^k)(b_l x^l)$
to get x^2-terms
we need $k+l=2$

x^3: $\cdots$

$\vdots$

Hence $f(x) g(x) = a_0 b_0 + (a_0 b_1 + a_1 b_0) x + \cdots$

In general,

$$f(x) g(x) = \sum_{n=0}^{\infty} c_n x^n$$

where $c_n = \underbrace{\sum_{k=0}^{n} a_k b_{n-k}}$.

want
$(a_k x^k) \cdot (b_l x^l)$
to give
x^n-term
$\Rightarrow$
$k+l=n$
$\Rightarrow l = n-k$
since $l \geq 0$, $\Rightarrow k \leq n$

nth - Cauchy product

$$\star \begin{cases} n=0 \Rightarrow c_0 = a_0 b_0 \\ n=1 \Rightarrow c_1 = a_1 b_0 + a_0 b_1 \\ n=2 \Rightarrow c_2 = a_2 b_0 + a_1 b_1 + a_0 b_2 \\ n=3 \Rightarrow c_3 = a_3 b_0 + a_2 b_1 + a_1 b_2 + a_0 b_3 \end{cases}$$

- Differentiation of a power series

Assume $f(x) = \sum_{n=0}^{\infty} a_n x^n$ on $I = (-r, r)$

$$\Rightarrow \quad f'(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right)'$$

will see why in Math 142/140

$$= \sum_{n=0}^{\infty} (a_n x^n)'$$

$$= \sum_{n=0}^{\infty} a_n n x^{n-1}$$

$$= \sum_{n=1}^{\infty} n a_n x^{n-1}$$

why: $n=0 \Rightarrow a_n \cdot n \cdot x^{n-1} = a_0 \cdot 0 \cdot x^{0-1} = 0$

holds on I

• Integration of a power series

Assume $f(x)=\sum_{n=0}^{\infty} a_n x^n$ on $I=(-r,r)$.

Then on I,

$$\int f(x)\,dx = \int \left(\sum_{n=0}^{\infty} a_n x^n\right) dx$$

$$= \sum_{n=0}^{\infty} \underbrace{\int a_n x^n\,dx}$$

$$= \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} + C$$

by Math 142/140

and

$$\int_a^b f(x)\,dx = \int_a^b \left(\sum_{n=0}^{\infty} a_n x^n\right) dx$$

$$= \sum_{n=0}^{\infty} \int_a^b a_n x^n\,dx$$

for $a, b \in I$.

☆ E.g: Given $\underbrace{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n}_{\text{known}}$ on $(-1,1)$

Compute the power series of

(a) $\frac{1}{1+x^2}$

(b) $\frac{1}{(x-1)^2}$

(c) $\arctan x$

Derive new series from a known series

A: (a). We use substitution method.

Note: $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$

Given $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $|x| < 1$

Replace/substitute all "x" by "$-x^2$"

$$\Rightarrow \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n, \quad |-x^2| < 1$$

$$\Rightarrow \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad \underline{|x|^2 < 1}$$

$\Leftrightarrow |x| < 1$

$$\Rightarrow \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad |x| < 1$$

(b) Note $\frac{1}{(x-1)^2} = \left(\frac{1}{1-x}\right)'$

Recall if P10

$f(x) = \sum_{n=0}^{\infty} a_n x^n$ on I

$\Rightarrow$

$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ on I

Given $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ on $(-1, 1)$

Differentiate both sides $\Rightarrow$

$$\frac{1}{(x-1)^2} = \left(\frac{1}{1-x}\right)' = \sum_{n=1}^{\infty} n x^{n-1} \text{ on } (-1,1)$$

(c)

Note $\arctan x = \int_0^x \frac{1}{1+t^2} dt$

By (a), $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ on $(-1, 1)$

$$\Rightarrow \quad \frac{1}{1+t^2} = \sum_{n=0}^{\infty} (-1)^n t^{2n} \quad \text{on } (-1, 1)$$

Recall if P11

$f(t) = \sum_{n=0}^{\infty} a_n t^n$ on I

$\Rightarrow$

$\int_a^b f(t) dt = \sum_{n=0}^{\infty} \int_a^b a_n t^n dt$ on I

Let $x \in (-1, 1)$.

Integrating above from 0 to x

$\Rightarrow$

$$\int_0^x \frac{1}{1+t^2} dt = \sum_{n=0}^{\infty} \int_0^x (-1)^n t^{2n} dt$$

$$\Rightarrow \quad \arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{on } (-1,1)$$

Finally, we talk about "shifting the summation index".

E.g. Express the series

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

as a series where the general term is x^k.

This means we need to write it as

$$\sum_{k=?}^{\infty} ?\, x^k$$

?: to be calculated.

A: Let $k = n-2$. Then $n = k+2$.

In the expression "$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$", replace all "n" by "k+2". $\Rightarrow$

Warning:
Always/only replace terms by what they equal to

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

$$= \sum_{k+2=2}^{\infty} (k+2)(k+2-1)a_{k+2} x^{k+2-2}$$

$$= \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2} x^{k}$$